Exam Calculus of Variations and Optimal Control 2014-15

Date : 27-01-2015
Place : 5161.0165
Time : 14.00-17.00

The exam is OPEN BOOK; you can use all your books/papers/notes.
You are supposed to provide arguments to all your answers, and to explcitly refer to theorems/propositions whenever you use them.

1. Consider the scalar system

   $$\dot{x} = x + u, \quad x(0) = x_0$$

   The aim is to minimize the cost criterion

   $$\int_0^2 [-2x(t) + 3u(t) + \alpha u^2(t)]dt$$

   (a) Determine the Hamilton function of the Minimum Principle, and derive the differential equation and boundary condition(s) for the co-state p.

   (b) Show that the evolution of the co-state is given by

   $$p(t) = -2e^{(2-t)} + 2, \quad t \in [0, 2]$$

   (c) Take $\alpha = \frac{1}{2}$, and determine for this case the optimal input $u : [0, 2] \to \mathbb{R}$.

   (d) Now consider the case $\alpha = 0$, and moreover assume that u satisfies the constraints $0 \leq u \leq 2$. Determine the optimal input function $u : [0, 2] \to \mathbb{R}$.

2. Consider the scalar system

   $$\dot{x} = -x + u, \quad x(0) = x_0,$$

   together with the cost criterion

   $$\int_0^1 [(x(t) - c)^2 + u^2(t)]dt$$

   for a certain constant c.

   (a) Take $c = 0$, and solve the optimal control problem by dynamic programming.

   (b) Take $c = 1$. Determine the Hamilton-Jacobi-Bellman equation in the (unknown) value function $V(x, t)$. Take a candidate value function of the form

   $$V(x, t) = p(t)x^2 + q(t)x + r(t)$$

   for certain function p, q, r. Argue, without going into computational details, how this will lead to a solution of the optimal control problem.

3. Consider the system

$$\frac{d}{dt}y = \alpha y - u, \quad y(0) = y_0, \tag{1}$$

with $\alpha \in \mathbb{R}$ an unknown parameter.

(a) Show that the feedback $u = ky$ for $k \in \mathbb{R}$ sufficiently large renders the equilibrium $y = 0$ asymptotically stable.

Since α is unknown we do not know how large we should take k. To solve this problem we let k to be determined by the following dynamics

$$\frac{d}{dt}k = y^2 - k, \quad k(0) = k_0, \tag{2}$$

for some k_0. Here the term y^2 ensures that k grows if y^2 is large, while $-k$ prohibits too large values of k

(b) By substituting $u = ky$ in the differential equation (1), one obtains the following coupled differential equations in y and k:

$$\begin{aligned} \tfrac{d}{dt}y &= \alpha y - ky \\ \tfrac{d}{dt}k &= y^2 - k \end{aligned} \tag{3}$$

Determine all equilibria $(\bar{y}, \bar{k})$ of (3). Make a distinction between the case $\alpha > 0$ and $\alpha \leq 0$.

(c) Investigate by means of linearization the stability of all equilibria found under (a). Distinguish again between the case $\alpha > 0$ and $\alpha \leq 0$.

(d) Take $\alpha < 0$. Show by means of a suitable Lyapunov function that for all initial values (y_0, k_0):

$$\lim_{t \to \infty} (y(t), k(t)) = (0, 0) \tag{4}$$

(e) Does (4) remain too hold for the case $\alpha = 0$?

4. Variational calculus can be extended to functions of more than one independent variable. In particular it can be used for functions depending on time $t \in \mathbb{R}$ *and* on a spatial variable $z \in \mathbb{R}$.

Consider thus functions

$$y : [t_1, t_2] \times [z_1, z_2] \to \mathbb{R},$$

also denoted by $y(t, z)$. Denote the partial derivative of $y(t, z)$ with respect to t by $y_t(t, z)$, and the partial derivative with respect to z by $y_z(t, z)$. Now consider the problem of minimizing the expression (for some function L)

$$\int_{t_1}^{t_2} \int_{z_1}^{z_2} L(y(t, z), y_t(t, z), y_z(t, z)) dz dt$$

over all functions $y : [t_1, t_2] \times [z_1, z_2] \to \mathbb{R}$ satisfying

$$y(t_1, z) = a_1,\ y(t_2, z) = a_2, \quad z \in [z_1, z_2]$$
$$y(t, z_1) = b_1,\ y(t, z_2) = b_2, \quad t \in [t_1, t_2]$$

(a) Show that the Euler-Lagrange equation for this case is given as the partial differential equation

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial y_t} - \frac{d}{dz}\frac{\partial L}{\partial y_z} = 0, \quad t \in [t_1, t_2], z \in [z_1, z_2]$$

(**Hint**: Define, similar to the case of one independent variable, variations $\alpha \delta y(t, z)$ with $\delta y(t, z)$ zero on the boundary of the rectangle $[t_1, t_2] \times [z_1, z_2]$. Now perform integration by parts *both* to the time variable t and to the spatial variable z.)

(b) Consider L given by

$$L(y(t, z), y_t(t, z), y_z(t, z)) = \frac{1}{2}(y_t^2(t, z) - y_z^2(t, z))$$

(This is the Lagrangian - kinetic energy minus potential energy- of a vibrating longitudinal string.) Derive the Euler-Lagrange equations for this case.

Distribution of points: Total 100; Free 10,

1. a: 10, b: 5, c: 5, d: 5.
2. a: 10, b: 10.
3. a: 5, b: 5, c: 10, d: 5, e: 5.
4. a: 10, b: 5.